

Positivity of line bundles in derived categories

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Motivation

Using the theory of Matsui spectra, the reconstruction theorem of Bondal-Orlov was recently generalized to the following setting.

Definition 1 A line bundle \mathcal{L} is said to be \otimes -**ample** if $\langle \mathcal{L}^{\otimes n} \mid n \in \mathbb{Z} \rangle = \text{Perf } X$.

On a quasi-projective variety, any (anti-)ample line bundle is \otimes -ample.

Theorem 2 (Ito-Matsui, Ito) Let X be a Gorenstein proper variety with \otimes -ample canonical bundle ω_X . Then, the following assertions hold:

- X can be reconstructed from the triangulated category structure of $\text{Perf } X$;
- If we have $\text{Perf } X \simeq \text{Perf } Y$ with a variety Y , then $X \cong Y$.

Here, note we do not require any projectivity, which is more natural from a dg-categorical perspective. Now, a natural question is whether or not this is actually a generalization.

Question 3 Are there \otimes -ample line bundles that are neither ample nor anti-ample?

Main Theorem (Ito-Olander) Let X be a proper variety and \mathcal{L} a line bundle. TFAE:

1. \mathcal{L} is \otimes -ample.
2. $\mathcal{L}|_Z$ is big or anti-big for every closed integral subscheme $Z \subset X$.

The theorem provides a variety of examples and well-behaved theory of \otimes -ample line bundles.

Big line bundles and affine complements

A key technical ingredient is the following characterization of big line bundles, which is interesting in its own right in relation to divisors with affine complements.

Proposition 4 Let X be an integral qcqs scheme and \mathcal{L} a line bundle. TFAE:

1. There exists an integer $n > 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the open $X_s := \{s \neq 0\} = X \setminus V(s)$ is non-empty and quasi-affine.
2. There exists an integer $n > 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the open X_s is non-empty and affine.
3. There exists an integer $n > 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the open X_s is non-empty and there exists an integral domain R and a morphism

$$X_s \rightarrow \text{Spec}(R)$$

whose generic fiber $(X_s)_K$ is quasi-affine where $K = \text{Frac}(R)$.

If X is a proper variety, then they are further equivalent to:

4. \mathcal{L} is big, i.e., there exist constants $m_0, C > 0$ such that $\dim_k \Gamma(X, \mathcal{L}^{\otimes m_0 m}) > C \cdot m^{\dim X}$ for any $m \gg 0$.

Definition 5 Let X be an integral qcqs scheme. A line bundle \mathcal{L} is **big** if the equivalent conditions 1-3 hold.

Our main theorem indeed holds in the generality of noetherian schemes with this notion of big line bundles. A proof of the main theorem goes as follows: (1 \Rightarrow 2) It is easy to see \otimes -amplitude is preserved under quasi-affine pullbacks, so it suffices to show a \otimes -ample line bundle on an integral closed subscheme is big or anti-big. Indeed, \otimes -amplitude provides a desired section. (2 \Rightarrow 1) We can show if there exists $n \neq 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is quasi-affine and $\mathcal{L}|_{V(s^r)}$ is \otimes -ample for each $r \geq 0$, then \mathcal{L} is \otimes -ample. Now we can conclude by noetherian induction.

Consequences of the main theorem

Interestingly we have not found a direct proof to the following fundamental facts.

Lemma 6 Let \mathcal{L} be a line bundle on a noetherian scheme X . Let $0 \neq n \in \mathbb{Z}$. Let $f : Y \rightarrow X$ be a finite surjective morphism of schemes.

1. \mathcal{L} is \otimes -ample if and only if $\mathcal{L}^{\otimes n}$ is \otimes -ample.
2. \mathcal{L} is \otimes -ample if and only if the restriction of \mathcal{L} to the irreducible components of X (with the reduced subscheme structure) are \otimes -ample.
3. \mathcal{L} is \otimes -ample if and only if $f^* \mathcal{L}$ is \otimes -ample.

In practice, the following seems to be a useful way to check if a line bundle is \otimes -ample.

Lemma 7 Let X be a noetherian scheme and \mathcal{L} a line bundle on X . Let $n_1, \dots, n_k \in \mathbb{Z}$ be integers and $s_i \in \Gamma(X, \mathcal{L}^{\otimes n_i})$ global sections. Assume:

1. X_{s_i} have \otimes -ample structure sheaf (for example if each X_{s_i} is quasi-affine).
2. The restriction of \mathcal{L} to the reduced closed subscheme $V(s_1, \dots, s_k)_{\text{red}} \subset X$ cut out by s_1, \dots, s_k is \otimes -ample.

Then \mathcal{L} is \otimes -ample.

\otimes -ample cone in $N^1(X)$

Our main theorem allows numerical studies of \otimes -ample line bundles.

Definition 8 Let X be a proper variety. We say an \mathbb{R} -Cartier divisor on X is \otimes -**ample** if its restriction to any closed subvariety is linearly equivalent to a big or anti-big \mathbb{R} -Cartier divisor. Let

$$\otimes\text{-Amp}(X) \subset N^1(X) = \text{Div}(X)_{\mathbb{R}} / \equiv_{\text{num}}$$

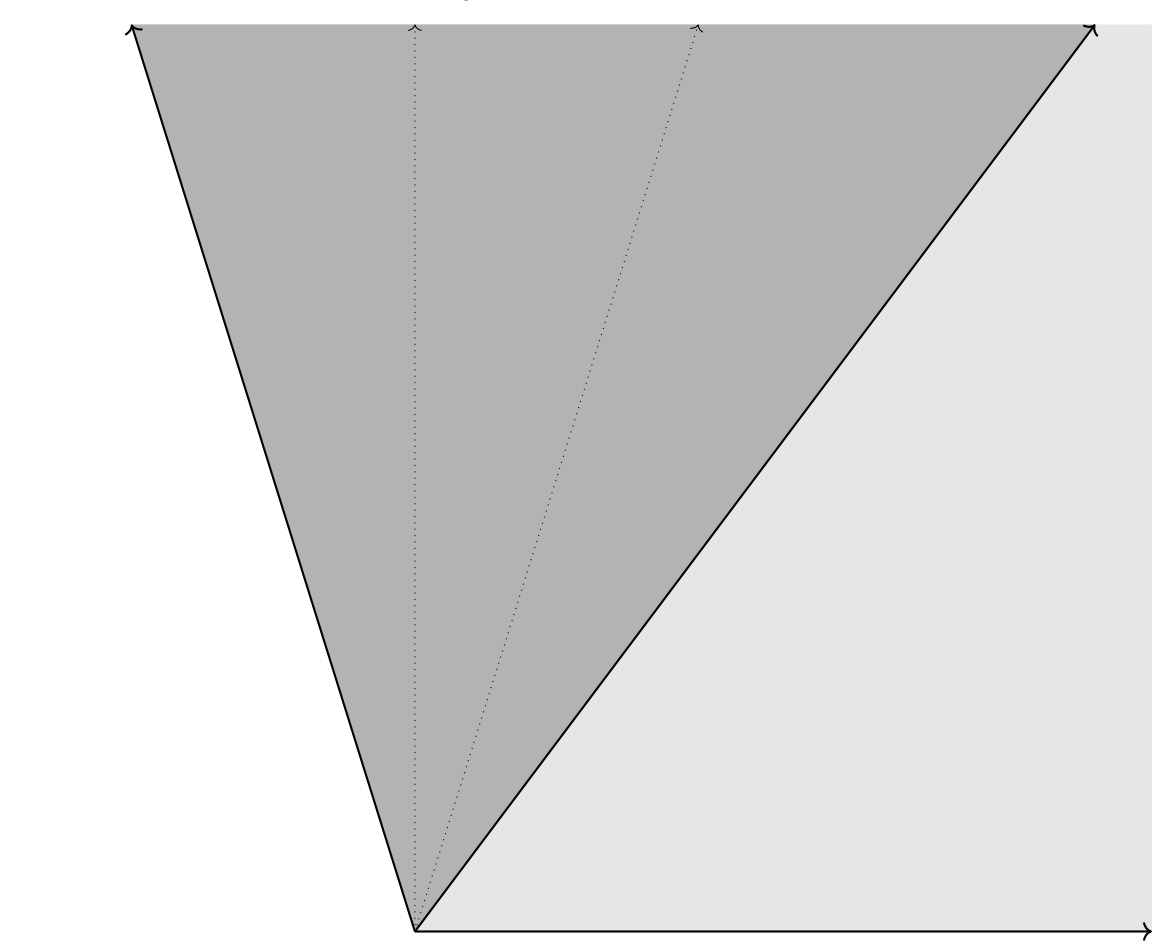
denote the cone of \otimes -ample \mathbb{R} -Cartier divisors up to numerical equivalence.

Lemma 9 Let X be a proper variety.

- A Cartier divisor is \otimes -ample if and only if its numerical class lies in $\otimes\text{-Amp}(X)$.
- $\otimes\text{-Amp}(X) \cap \text{Nef}(X) = \text{Amp}(X)$. In particular, $\otimes\text{-Amp}(X) \cap \partial \text{Nef}(X) = \emptyset$.

Note $\otimes\text{-Amp}(X)$ is indeed computable. Let $\pi : X = \mathbb{P}_X(\mathcal{E}) \rightarrow C$ be a ruled surface over a projective curve C . Suppose \mathcal{E} is unstable with a destabilizing quotient $\mathcal{E} \twoheadrightarrow \mathcal{Q}$, which corresponds to a section $C_0 \subset X$. Then, we get a complete description of cones in $N^1(X)$.

$$\mathbb{R}[C_0] \quad \mathbb{R}_+ \xi \quad \mathbb{R}_+[-K_X] \quad \mathbb{R}_+(\xi - \mu f)$$



$$f = [\text{fiber of } \pi], \xi = [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)] \in N^1(X)$$

$$\mu = \deg \mathcal{Q}, d = \deg \mathcal{E}, g = g(C)$$

$$[C_0] = \xi + (\mu - d)f, [K_X] = -2\xi + (2g - 2 + d)f$$

$$\text{Amp}(X)$$

$$\otimes\text{-Amp}_+(X) \setminus \text{Amp}(X)$$

$$\text{Nef}(X) = \langle \mathbb{R}_+(\xi - \mu f), \mathbb{R}_+ f \rangle$$

$$\text{PsEff}(X) = \langle \mathbb{R}_+[C_0], \mathbb{R}_+ f \rangle$$

Here, $\otimes\text{-Amp}_+(X) = \otimes\text{-Amp}(X) \cap \text{Big}(X)$.

Examples of \otimes -ample line bundles

The following completely answer Question 3 for smooth projective surfaces.

Lemma 10 A line bundle \mathcal{L} on a proper surface is \otimes -ample if and only if $\deg \mathcal{L}|_C \neq 0$ for every integral closed curve $C \subset X$ with $C^2 < 0$. Moreover,

$$\otimes\text{-Amp}_+(X) = \text{Big}(X) \setminus \bigcup \{C^\perp : C \subset X \text{ an integral curve with } C^2 < 0\}.$$

Lemma 11 A smooth projective surface X has a \otimes -ample but neither ample nor anti-ample line bundle if and only if there is an integral curve $C \subset X$ with $C^2 < 0$.

Other types of examples include but not limited to the following.

- Let X be a quasi-projective variety and let $\pi : X \rightarrow Y$ be the blow-up at finitely many points with corresponding exceptional divisors E_i . Then for any ample line bundle \mathcal{L} and for any $l_i > 0$, $\pi^* \mathcal{L} \otimes \bigotimes \mathcal{O}_X(l_i E_i)$ is neither ample nor anti-ample, but \otimes -ample.
- Let $X = \text{Bl}_{(0,0)} \mathbb{A}_k^2 \setminus \{p\}$ where p is a k -point of an exceptional divisor. Then the structure sheaf \mathcal{O}_X is \otimes -ample but not ample. In particular, being quasi-projective and having a \otimes -ample structure sheaf do not imply being quasi-affine.
- The affine space with doubled origin has the \otimes -ample structure sheaf. In particular, having a \otimes -ample line bundle does not imply neither separated nor having a resolution property.
- If X is a union of two copies of \mathbf{P}^1 glued along a node and \mathcal{L} is obtained by gluing $\mathcal{O}(1)$ on one copy with $\mathcal{O}(-1)$ on the other, then \mathcal{L} is \otimes -ample.
- Hironaka's example of a non-projective proper variety has a \otimes -ample line bundle.

Examples of \otimes -ample canonical bundles

Some of the previous examples provide varieties with \otimes -ample (but neither ample nor anti-ample) canonical bundle, to which we can apply Theorem 2.

- For a smooth projective surface X , ω_X is \otimes -ample if and only if ω_X is big or anti-big and X contains no (-2) -curve. For example, a smooth projective toric surface with no (-2) -curve, a projective bundle of an unstable rank 2 vector bundle over an elliptic curve, and the blow-up of \mathbb{P}^2 at r points on a line with $r \neq 3$ have \otimes -ample canonical bundles.
- If X is a quasi-projective variety with ample canonical bundle (e.g. a quasi-affine variety), then its blow-up at finitely many closed points has a \otimes -ample canonical bundle.
- Take the Fermat hypersurface $X = \{x_0^d + \dots + x_4^d = 0\} \subset \mathbb{P}_k^4$ with odd $d > 5$. Then, there is a line $l \subset X$ with normal bundle $\mathcal{O}_l(1) \oplus \mathcal{O}_l(2 - d)$. Then the blow-up of X along the line l has a \otimes -ample canonical bundle.
- A proper toric variety has a \otimes -ample canonical bundle if and only if the restriction of the canonical line bundle to the torus boundary divisor is \otimes -ample.

References

- D. Ito and H. Matsui, A new proof of the Bondal-Orlov reconstruction theorem using Matsui spectra (to appear in Bull. Lond. Math. Soc.)
- D. Ito, Polarizations on a triangulated category (available at arXiv:2502.15621)
- D. Ito and N. Olander, On \otimes -ample line bundles (coming soon)