Geometric Construction of Quiver Tensor Products

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Motivation

Balmer's tensor triangular geometry shows that the data of an algebraic variety X in the derived category $\mathsf{Perf}(X)$ of **perfect complexes** (complexes locally quasi-isomorphic to bounded complexes of vector bundles) is fully encoded in the monoidal structure $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$.

Ambitious Question Can we characterize the monoidal structure $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$ in the "moduli space" of monoidal structures on $\mathsf{Perf}(X)$?

A more tractable but already interesting first question is:

Question What kinds of monoidal structures does $\mathsf{Perf}(\mathbb{P}^n)$ have?

We will see that for any finite-dimensional algebra A, there is a "convolution-like" monoidal structure $\star'_A$ on $\mathsf{Perf}(\mathbb{P}(A))$ and this monoidal structure reconstructs A.

Quiver tensor product on $\mathsf{Perf}(\mathbb{P}^n)$

Beilinson showed that the derived category $\mathsf{Perf}(\mathbb{P}^n)$ is equivalent to the bounded derived category $\mathsf{D}^b\mathsf{rep}(\mathsf{Beil}_n)$ of finite-dimensional quiver representations, where

$$\mathsf{Beil}_n = \overset{0}{\bullet} \;\begin{array}{c} -x_0\to \\ -x_1\to \\ \cdots\cdots \\ x_{n-1} \\ -x_n\to \end{array}\; \overset{1}{\bullet} \;\begin{array}{c} -x_0\to \\ -x_1\to \\ \cdots\cdots \\ x_{n-1} \\ -x_n\to \end{array}\; \cdots \;\begin{array}{c} -x_0\to \\ -x_1\to \\ \cdots\cdots \\ x_{n-1} \\ -x_n\to \end{array}\; \overset{n-1}{\bullet} \;\begin{array}{c} -x_0\to \\ -x_1\to \\ \cdots\cdots \\ x_{n-1} \\ -x_n\to \end{array}\; \overset{n}{\bullet}$$

with relation $x_ix_j = x_jx_i$. Under the equivalence $\mathsf{Perf}(\mathbb{P}^n) \simeq \mathsf{D}^b\mathsf{rep}(\mathsf{Beil}_n)$, $\mathsf{Perf}(\mathbb{P}^n)$ admits a monoidal structure $\otimes_{\mathsf{quiv}}$ corresponding to the vertex-wise tensor product of quiver representations on Beil_n. Let's see what this means through the examples below!

Examples

Let's look at $\mathbb{P}^1$ for simplicity. Beil_1 is the Kronecker quiver

$$\bullet \rightrightarrows \bullet$$

We can define a tensor product of two quiver representations

$$V = \left(V_0 \underset{-f_1\to}{\overset{-f_0\to}{}} V_1 \right), \quad W = \left(W_0 \underset{-g_1\to}{\overset{-g_0\to}{}} W_1 \right)$$

by setting

$$V \otimes_{\mathsf{quiv}} W := \left(V_0 \otimes W_0 \underset{-f_1\otimes g_1\to}{\overset{-f_0\otimes g_0\to}{}} V_1 \otimes W_1 \right).$$

This defines a monoidal category $(\mathsf{Perf}\,\mathbb{P}^1, \otimes_{\mathsf{quiv}})$. Let's try to understand $\otimes_{\mathsf{quiv}}$ through some examples! Under the equivalence $\mathsf{Perf}\,\mathbb{P}^1 \simeq \mathsf{D}^b\,\mathsf{rep}(\mathrm{Beil}_1)$, we have

$$k([x_0 : x_1]) \leftrightarrow \left(k \underset{-x_1\to}{\overset{-x_0\to}{}} k \right), \; \mathcal{O}_{\mathbb{P}^1} \leftrightarrow \left(0 \rightrightarrows k \right).$$

Thus, the unit of $\otimes_{\mathsf{quiv}}$ is $k([1:1])$ and we have

$$k([x_0 : x_1]) \otimes_{\mathsf{quiv}} k([y_0 : y_1]) = k([x_0y_0 : x_1y_1])$$

when the RHS makes sense. On the other hand,

$$k([1:0]) \otimes_{\mathsf{quiv}} k([0:1]) = \left(k \underset{-0\to}{\overset{-0\to}{}} k \right) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)[1].$$

Question Can we give a geometric understanding to $\otimes_{\mathsf{quiv}}$?

Convolutions on monoid schemes/stacks

Let X be a (nice) monoid scheme/stack with multiplication map $\mu : X \times X \to X$. Then, the derived category $\mathsf{D}_{\mathsf{qc}}(X)$ of quasi-coherent sheaves gets a **convolution tensor product** $\star$:

$$\mathsf{D}_{\mathsf{qc}}(X) \times \mathsf{D}_{\mathsf{qc}}(X) \ni (\mathcal{F}, \mathcal{G}) \mapsto \mathbb{R}\mu_*(p_1^*\mathcal{F} \otimes^{\mathbb{L}}_{X\times X} p_2^*\mathcal{G}) =: \mathcal{F} \star_X \mathcal{G} \in \mathsf{D}_{\mathsf{qc}}(X)$$

where $p_1, p_2 : X \times X \to X$ are projections. For example, if we consider $X = \mathbb{G}_m$, then

$$k(x) \star_{\mathbb{G}_m} k(y) = k(x \cdot y)$$

for any $x, y \in \mathbb{G}_m(k) = k^\times$. In particular, the quiver tensor product $\otimes_{\mathsf{quiv}}$ on $\mathbb{P}^1$ agrees with the convolution tensor product $\star_{\mathbb{G}_m}$ on $\mathbb{G}_m$ for skyscraper sheaves on $\mathbb{G}_m \subset \mathbb{P}^1$.

Although the monoid structure on $\mathbb{G}_m$ does not extend to $\mathbb{P}^1$, the convolution tensor product $\star_{\mathbb{G}_m}$ on $\mathbb{G}_m$ does extend to the monoidal structure $\otimes_{\mathsf{quiv}}$ on $\mathbb{P}^1$ on the categorical level!

Quiver tensor product is extended convolution product

Consider the graph $Z_\mu \subset (\mathbb{G}_m^n \times \mathbb{G}_m^n) \times \mathbb{G}_m^n$ of the multiplication $\mu : \mathbb{G}_m^n \times \mathbb{G}_m^n \to \mathbb{G}_m^n$. Then, the convolution product can also be written as

$$\mathcal{F} \star_{\mathbb{G}_m^n} \mathcal{G} = \mathbb{R}p_{3*}(p_1^*\mathcal{F} \otimes^{\mathbb{L}} p_2^*\mathcal{G} \otimes^{\mathbb{L}} \mathcal{O}_{Z_\mu})$$

where $p_1, p_2, p_3 : (\mathbb{G}_m^n \times \mathbb{G}_m^n) \times \mathbb{G}_m^n \to \mathbb{G}_m^n$ are projections. Now, letting $\bar{Z}_\mu$ denote the closure of Z_μ in $(\mathbb{P}^n \times \mathbb{P}^n) \times \mathbb{P}^n$ and p_i denote the projections, we claim:

Theorem 1 (Ito–Nolan) *For $\mathcal{F}, \mathcal{G} \in \mathsf{Perf}(\mathbb{P}^n)$, there is a natural isomorphism*

$$\mathcal{F} \otimes_{\mathsf{quiv}} \mathcal{G} \cong \mathbb{R}p_{3*}(p_1^*\mathcal{F} \otimes^{\mathbb{L}} p_2^*\mathcal{G} \otimes^{\mathbb{L}} \mathcal{O}_{\bar{Z}_\mu}).$$

In this sense, the quiver tensor product is the extended/compactified convolution product. Note that it is not obvious why the right hand side indeed satisfies compatibilities to define a monoidal structure!

Window–Hitchcock adjunction and main results

For our proof, we utilize the window theory developed by Halpern-Leistner and Ballard–Favero–Katzarkov. Recall that we can present

$$i : \mathbb{P}^n = [(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m] \subset [\mathbb{A}^{n+1}/\mathbb{G}_m] =: \mathfrak{X}.$$

There is a fully faithful embedding $W : \mathsf{D}_{\mathsf{qc}}(\mathbb{P}^n) \hookrightarrow \mathsf{D}_{\mathsf{qc}}(\mathfrak{X})$, called the **window functor**, that restricts to a fully faithful embedding

$$\mathsf{Perf}\,\mathbb{P}^n = \langle \mathcal{O}_{\mathbb{P}^n}, \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle \simeq \langle \mathcal{O}_{\mathfrak{X}}, \ldots, \mathcal{O}_{\mathfrak{X}}(n) \rangle \subset \mathsf{Perf}\,\mathfrak{X}.$$

We call the left adjoint $H : \mathsf{Perf}(\mathfrak{X}) \to \mathsf{Perf}(\mathbb{P}^n)$ the **Hitchcock functor**.

Now, note that given a finite-dimensional k-algebra A, the quotient stack $[\mathbb{A}(A)/\mathbb{G}_m]$ gets a monoid structure and hence $\mathsf{D}_{\mathsf{qc}}([\mathbb{A}(A)/\mathbb{G}_m])$ gets a convolution tensor product $\star_A$. For example, if we set $A = k^{n+1}$ with usual multiplication, $\mathfrak{X} = [\mathbb{A}^{n+1}/\mathbb{G}_m]$ gets the coordinate-wise multiplication. Our theorem says:

Main Theorem (Ito–Nolan) Let A be a finite-dimensional (commutative) k-algebra.

- There exists a unique (symmetric) monoidal structure $\star'_A$ on $\mathsf{Perf}(\mathbb{P}(A))$ such that the Hitchcock functor is (symmetric) monoidal with respect to the convolution product $\star_A$.
- For $\mathcal{F}, \mathcal{G} \in \mathsf{Perf}(\mathbb{P}(A))$, there is a natural isomorphism
$$\mathcal{F} \star'_A \mathcal{G} \cong \mathbb{R}p_{3*}(p_1^*\mathcal{F} \otimes^{\mathbb{L}} p_2^*\mathcal{G} \otimes^{\mathbb{L}} \mathcal{O}_{\bar{Z}_\mu})$$
where $\bar{Z}_\mu$ denotes the graph closure of the multiplication $[\mathbb{A}(A)^\times/\mathbb{G}_m] \times [\mathbb{A}(A)^\times/\mathbb{G}_m] \to [\mathbb{A}(A)^\times/\mathbb{G}_m]$ in $(\mathbb{P}(A) \times \mathbb{P}(A)) \times \mathbb{P}(A)$.
- The embedding $\mathbb{R}j_* : (\mathsf{D}_{\mathsf{qc}}([\mathbb{A}(A)^\times/\mathbb{G}_m]), \star_{[\mathbb{A}(A)^\times/\mathbb{G}_m]}) \hookrightarrow (\mathsf{D}_{\mathsf{qc}}(\mathbb{P}(A)), \star'_A)$ is (symmetric) monoidal.
- The construction $A \mapsto \star'_A$ is functorial on surjective maps and essentially injective.

Moreover, we proved that $\star'_{k^{n+1}}$ coincides with $\otimes_{\mathsf{quiv}}$ which proves the previous theorem. In this case, we have $[(\mathbb{A}^{n+1})^\times/\mathbb{G}_m] = \mathbb{G}_m^n$ and we may view $(\mathsf{D}_{\mathsf{qc}}(\mathbb{P}(A)), \star'_A)$ as a **categorical compactification** of $\mathbb{G}_m^n$ in $[\mathbb{A}^{n+1}/\mathbb{G}_m]$ by the third claim. In general, the group scheme $[\mathbb{A}(A)^\times/\mathbb{G}_m]$ may be isomorphic to $\mathbb{G}_a$, GL_n, PGL_n, for instance.

Ideas of proofs

In our proof, we developed a general framework using higher algebras and derived algebraic geometry. In particular, our statements hold in the language of dg-/∞-categories and $\mathbb{E}_n$-monoidal structures. Let us comment on several ideas in our proofs.

- The first claim of the main theorem follows from the categorical properties of the window functor being fully faithful, preserving compact objects and having the essential image compatible with $\star_A$ in a certain way. This is a stable version of a claim in Lurie's Higher Algebra.
- The second claim follows from the direct computations of the Fourier–Mukai kernel for the window functor, using an idea similar to a resolution of the diagonal, together with yoga of base change, projection formula, and the window–Hitchcock adjunction.
- To show the equivalence of $\star'_{k^{n+1}}$ and $\otimes_{\mathsf{quiv}}$, we first show that there is an equivalence
$$(\mathsf{D}_{\mathsf{qc}}(\mathfrak{X}), \star_{\mathsf{conv}}) \simeq (\mathsf{DRep}(\mathsf{Beil}_n^\infty), \otimes_{\mathsf{quiv}})$$
where Beil_n^∞ is the quiver obtained by infinitely extending Beil_n both to the left and to the right. The construction of an equivalence of the underlying categories is straightforward, but to show the equivalence is monoidal, we make use of the monoidal adjunction of Toën–Vaquié
$$(\mathsf{Perf}^*)^{\mathsf{op}} : (\mathsf{dStk}_k, \times) \rightleftarrows \left((\mathsf{Cat}_k^{\mathsf{perf}})^{\mathsf{op}}, \otimes_k\right) : \mathfrak{Rep}$$
between derived stacks over k and perfect k-linear ∞-categories.

Implications to Homological Mirror Symmetry

Our main results extend to any **smooth complete toric variety of Bondal–Ruan type**, including many toric Fano varieties. For such a variety X, the Bondal–Thomsen collection Θ gives a full strong exceptional collection of line bundles on X, and hence we have an equivalence $\mathsf{Perf}(X) \simeq \mathsf{D}^b\mathsf{rep}(\mathsf{Q}_\Theta)$ for a certain quiver Q_Θ with relations. We may understand the equivalence through homological mirror symmetry as follows: Let M be the cocharacter lattice of the dense torus in X and $M_{\mathbb{R}}/M$ a real torus of dimension $\dim X$. Then there is a natural equivalence

$$\mathsf{Perf}(X) \simeq \mathsf{D}^b\mathsf{rep}(\mathsf{Q}_\Theta) \simeq \mathsf{Sh}_Z^{\mathsf{perf}}(M_{\mathbb{R}}/M) \tag{1}$$

where $\mathsf{Sh}_Z^{\mathsf{perf}}(M_{\mathbb{R}}/M)$ denotes the derived category of certain constructible sheaves of k-vector spaces on the real torus $M_{\mathbb{R}}/M$. In particular, $\mathsf{Sh}_Z^{\mathsf{perf}}(M_{\mathbb{R}}/M)$ is closed under stalk-wise tensor product $\otimes_k$ of constructible sheaves. Building on a work of David Favero and Jesse Huang on homotopy path algebras, we show

Theorem 2 (Ito–Nolan) *For a smooth complete toric variety of Bondal–Ruan type,*

$$\left(\mathsf{Perf}(X), \star'_X\right) \simeq \left(\mathsf{D}^b\mathsf{rep}(\mathsf{Q}_\Theta), \otimes_{\mathsf{quiv}}\right) \simeq \left(\mathsf{Sh}_Z^{\mathsf{perf}}(M_{\mathbb{R}}/M), \otimes_k\right),$$

where $\star'_X$ is the extended convolution of the dense torus action on X.

References

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