TT-structures arising from quiver representations and extended convolution products on toric varieties

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Derived Representation Theory and Triangulated Categories
Aristotle University of Thessaloniki, June 26, 2025

Motivation

Balmer's tensor triangular geometry says information of an algebraic variety X in the perfect derived category Perf X is fully encoded in the tensor triangulated structure $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$.

Ambitious Question

Can we characterize $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$ in the "moduli space" of tt-structures on Perf X?

A much more tractable but already interesting first question is:

Question

What kinds of tt-structures does Perf $\mathbb{P}^n$ have?

We will see for any finite dimensional algebra A, there is a "convolution-like" tt-structure $\star_A$ on Perf $\mathbb{P}(A)$.

Derived quiver representations

We begin with a motivating case, where $A = k^{n+1}$ with coordinate-wise multiplication.

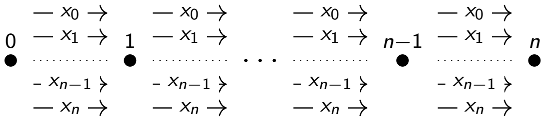
Theorem (Beilinson '78)

$\mathcal{T} = \mathcal{O}_{\mathbb{P}^n} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(n)$ is a tilting object in $\operatorname{Perf} \mathbb{P}^n$.

This allows us to view

$$\operatorname{Perf} \mathbb{P}^n \simeq \mathrm{D}^b \operatorname{mod}(\operatorname{End}(\mathcal{T})) \simeq \mathrm{D}^b \operatorname{rep}(\mathrm{Beil}_n),$$

where $\mathrm{D}^b \operatorname{Rep}(\mathrm{Beil}_n)$ is the bounded derived category of finite dimensional quiver representations on the n-Beilinson quiver

$\mathrm{Beil}_n =$ 

with relation $x_i x_j = x_j x_i$.

Quiver tensor product

Let's look at $\mathbb{P}^1$ for simplicity. Beil_1 is the Kronecker quiver

$$\bullet \rightrightarrows \bullet$$

We can define a tensor product of two quiver representations

$$V = \left(V_0 \overset{f_0}{\underset{f_1}{\rightrightarrows}} V_1 \right), \quad W = \left(W_0 \overset{g_0}{\underset{g_1}{\rightrightarrows}} W_1 \right)$$

by setting

$$V \otimes_{\mathrm{quiv}} W := \left(V_0 \otimes W_0 \overset{f_0 \otimes g_0}{\underset{f_1 \otimes g_1}{\rightrightarrows}} V_1 \otimes W_1 \right).$$

This defines a tt-category $(\mathrm{Perf}\,\mathbb{P}^1, \otimes_{\mathrm{quiv}})$. Let's try to understand $\otimes_{\mathrm{quiv}}$ through some examples!

Examples

Under the equivalence $\operatorname{Perf}\mathbb{P}^1 \simeq \mathrm{D}^b \operatorname{rep}(\mathrm{Beil}_1)$, we have

$$k([x_0 : x_1]) \leftrightarrow \left(k \underset{-x_1\to}{\overset{-x_0\to}{\rightrightarrows}} k \right), \ \mathcal{O}_{\mathbb{P}^1} \leftrightarrow \left(0 \rightrightarrows k \right).$$

Thus, the unit of $\otimes_{\mathrm{quiv}}$ is $k([1:1])$ and we have

$$k([x_0 : x_1]) \otimes_{\mathrm{quiv}} k([y_0 : y_1]) = k([x_0 y_0 : x_1 y_1])$$

when the RHS makes sense. On the other hand,

$$k([1:0]) \otimes_{\mathrm{quiv}} k([0:1]) = \left(k \underset{-0\to}{\overset{-0\to}{\rightrightarrows}} k \right) = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)[1].$$

Thus, although $\otimes_{\mathrm{quiv}}$ looks like a convolution, but it is not quite as $\mathbb{P}^n$ is not a monoid...

Window theory (very special case)

One way to fix this is to "embed" $\mathbb{P}^n$ into a monoid (stack). Recall we have $\mathbb{P}^n = [(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m] \subset [\mathbb{A}^{n+1}/\mathbb{G}_m] =: \mathfrak{X}$.

Theorem (Halpern-Leistner, Ballard-Favero-Katzarkov'12)

There is a fully faithful embedding $w : \mathrm{D}_{\mathrm{qc}}(\mathbb{P}^n) \hookrightarrow \mathrm{D}_{\mathrm{qc}}(\mathfrak{X})$, called the window, that restricts to a fully faithful embedding $\operatorname{Perf} \mathbb{P}^n = \langle \mathcal{O}_{\mathbb{P}^n}, \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle \simeq \langle \mathcal{O}_{\mathfrak{X}}, \ldots, \mathcal{O}_{\mathfrak{X}}(n) \rangle \subset \operatorname{Perf} \mathfrak{X}$.

Note $w \ncong \mathbb{L}i_*$ and $\mathrm{D}_{\mathrm{qc}}(\mathfrak{X})(\simeq \mathrm{D}\operatorname{Mod}^{\mathrm{gr}}(k[x_0, \ldots, x_n]))$ has the convolution tensor product (or the $\mathbb{Z}$-graded tensor product over k)

$$(\mathcal{F}, \mathcal{G}) \mapsto \mathbb{R}\mu_*(p_1^*\mathcal{F} \otimes^{\mathbb{L}}_{\mathfrak{X}\times\mathfrak{X}} p_2^*\mathcal{G}) =: \mathcal{F} \star_{\mathrm{conv}} \mathcal{G}$$

where $\mu, p_1, p_2 : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ are the coordinate-wise multiplication and projections, respectively.

Quiver tensor product on a "big quiver"

We define the "big" Beilinson quiver to be

$$\mathrm{Beil}_n^\infty = \cdots \overset{-1}{\bullet} \begin{matrix} \xrightarrow{x_0} \\ \xrightarrow{x_1} \\ \cdots \\ \xrightarrow{x_{n-1}} \\ \xrightarrow{x_n} \end{matrix} \overset{0}{\bullet} \begin{matrix} \xrightarrow{x_0} \\ \xrightarrow{x_1} \\ \cdots \\ \xrightarrow{x_{n-1}} \\ \xrightarrow{x_n} \end{matrix} \cdots \begin{matrix} \xrightarrow{x_0} \\ \xrightarrow{x_1} \\ \cdots \\ \xrightarrow{x_{n-1}} \\ \xrightarrow{x_n} \end{matrix} \overset{n}{\bullet} \begin{matrix} \xrightarrow{x_0} \\ \xrightarrow{x_1} \\ \cdots \\ \xrightarrow{x_{n-1}} \\ \xrightarrow{x_n} \end{matrix} \overset{n+1}{\bullet} \cdots$$

with relation $x_i x_j = x_j x_i$. Note $\mathrm{D}\,\mathrm{Rep}(\mathrm{Beil}_n^\infty)$ has $\otimes_{\mathrm{quiv}}$ as before.

Theorem (I.-Nolan)

There is a symmetric monoidal equivalence

$$(\mathrm{D}_{\mathrm{qc}}(\mathfrak{X}), \star_{\mathrm{conv}}) \simeq (\mathrm{D}\,\mathrm{Rep}(\mathrm{Beil}_n^\infty), \otimes_{\mathrm{quiv}}).$$

Hitchcock functor and main theorem

Proposition

The right adjoint of the window $w : \mathrm{D}_{\mathrm{qc}}(\mathbb{P}^n) \hookrightarrow \mathrm{D}_{\mathrm{qc}}(\mathfrak{X})$ *is given by the restriction of quiver representations along* $\mathrm{Beil}_n \subset \mathrm{Beil}_n^\infty$

$$H_w : \mathrm{D}_{\mathrm{qc}}(\mathfrak{X}) \simeq \mathrm{D}\,\mathrm{Rep}(\mathrm{Beil}_n^\infty) \to \mathrm{D}\,\mathrm{Rep}(\mathrm{Beil}_n) \simeq \mathrm{D}_{\mathrm{qc}}(\mathbb{P}^n)$$

We call H_w *the Hitchcock functor as it looks into* $\mathrm{D}_{\mathrm{qc}}(\mathbb{P}^n)$ *through the window.*

Theorem (I.-Nolan)

The quiver tensor product $\otimes_{\mathrm{quiv}}$ *on* $\mathrm{D}_{\mathrm{qc}}(\mathbb{P}^n)$ *is a unique symmetric monoidal structure such that the Hitchcock functor* $H_w : (\mathrm{D}_{\mathrm{qc}}(\mathfrak{X}), \star_{\mathrm{conv}}) \to (\mathrm{D}_{\mathrm{qc}}(\mathbb{P}^n), \otimes_{\mathrm{quiv}})$ *is symmetric monoidal.*

Extended convolutions and compactification

Our main theorem holds more generally (more general than below).

Theorem (I.-Nolan)

Let A be a finite-dimensional $\mathbb{E}_n$-algebra. Then, $\mathrm{D}_{\mathrm{qc}}(\mathbb{P}(A))$ *has a unique $\mathbb{E}_n$-monoidal structure $\star_A$ such that the Hitchcock functor is $\mathbb{E}_n$-monoidal. Moreover, $\star_A$ restricts to a tt-structure on* $\operatorname{Perf}\mathbb{P}(A)$.

Corollary

For any open submonoid $M \overset{i}{\hookrightarrow} \mathbb{P}(A) \subset [\mathbb{A}(A)/\mathbb{G}_m]$,
$\mathbb{R}i_* : (\mathrm{D}_{\mathrm{qc}}(M), \star_{\mathrm{conv}}) \hookrightarrow (\mathrm{D}_{\mathrm{qc}}(\mathbb{P}(A)), \otimes_A)$ *is $\mathbb{E}_n$-monoidal.*

In particular, we may think of $(\operatorname{Perf}(\mathbb{P}(A)), \star_A)$ as a "categorical compactification" of M in $[\mathbb{A}(A)/\mathbb{G}_m]$.

Examples and geometric remarks

Example

1. For $A = k^2$, $(\mathrm{Perf}\,\mathbb{P}^1, \star_{\mathrm{quiv}})$ compactifies $\mathbb{G}_m$.
2. For $A = k[\varepsilon]/\varepsilon^2$, $(\mathrm{Perf}\,\mathbb{P}^1, \star_A)$ compactifies $\mathbb{G}_a = \mathbb{A}^1$.
3. More generally, for a finite dimensional algebra B, $(\mathrm{Perf}\,\mathbb{P}(B \times k), \star_{B\times k})$ compactifies B.
4. For $A = \mathrm{Mat}_{n\times n}(k)$, $(\mathrm{Perf}\,\mathbb{P}^{n^2-1}, \star_A)$ "compactifies" $\mathrm{PGL}_n(k)$.

- This story works for any proper toric variety with a strong full exceptional collection of line bundles.
- If we know the resolution of diagonal for these toric varieties, we can explicitly compute the Fourier-Mukai kernel for the extended convolution product.

Thank you for your attention!