K3 surfaces

Torelli thm.

Thm (Global Trelli) weigh 2 Hodge

X, Y K3 surfaces.

$X \cong Y \iff H^2(X,\mathbb{Z}) \overset{\sim}{=} H^2(Y;\mathbb{Z})$

[presence intersection prod. $H^{2,0}(X) \mapsto H^{2,0}(Y)$]

$H^2(X,\mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$.

Question X, Y K3.

$D^b(X) \simeq D^b(Y) \overset{?}{\Rightarrow} X \cong Y? (\Leftarrow H^2(X,\mathbb{Z}) = H^2(Y,\mathbb{Z}))$

It's not true...

$((f_1, f_2, f_3), (g_1, g_2, g_3))$

Define $\tilde{H}^{1,1}(X) := H^{1,1} \oplus H^0 \oplus H^4$ $\quad \tilde{H}^{0,2}(X) = H^{0,2}(X)$

$\tilde{H}^{2,0}(X) := H^{2,0}(X)$.

$\rightarrow D^b(X) \simeq D^b(Y) \iff \tilde{H}(X) \overset{\text{Hodge}}{\cong} \tilde{H}(Y)$.

In studies of derived categories,
it is important to understand $\mathrm{Auteq}(D^b(X))$.

$\rightarrow$ There is an almost surjective map $\quad$ (0.1, 10.2

$\mathrm{Auteq}(D^b(X)) \rightarrow \mathrm{HodgeAut}(\tilde{H}(X))$. $\quad$ FM transform.

Moduli of stable bundles

X K3 surface and we fix some cohomological invt (such as chern character) of vectors called Mukai vector v.

$$= (rk, c_1, rk + c_1^2/2 - c_2) \in \overset{H^0}{\mathbb{Z}} \times H^2 \times \overset{H^4}{\mathbb{Z}}$$

Thm. A fine moduli of semi-stable v.b.s w/ assigned Mukai vector exists $\rightsquigarrow \mathcal{M}(X,v)$.

If v is good enough, $\mathcal{M}(X,v)$ is a K3 surface. ⌟

For AGers, there is a universal bundle $\mathcal{U}$ on $X \times \mathcal{M}(X,v)$

$$D^b(X) \xrightarrow{\sim} D^b(\mathcal{M}(X,v)) \quad (\Rightarrow \mathcal{M}(X,v) \text{ is a K3 surface})$$

$$\mathcal{F}^\bullet \longmapsto \Phi_{\mathcal{U}}(\mathcal{F}^\bullet) = p_{\mathcal{M}*}(p_X^*\mathcal{F}^\bullet \otimes_{X\times\mathcal{M}} \mathcal{U})$$

$$\begin{tikzcd} & X \times \mathcal{M}(X,v) \ni \boxed{\mathcal{U}} \arrow[dl, "p_X"'] \arrow[dr, "p_{\mathcal{M}}"] & \\ \mathcal{F}^\bullet \; X & & \mathcal{M}(X,v) \end{tikzcd}$$

Understand all K3 surfaces that are FM partners. (this is finite).

Stability conditions on derived categories.

For any smth proj. var. X, Y

if $\Phi : D^b(X) \xrightarrow{\sim} D^b(Y)$,

then $\exists\, \mathcal{F}^\bullet \in D^b(X \times Y)$ s.t. $\Phi \cong \Phi_{\mathcal{F}^\bullet}$

Rational $\overset{\text{smooth}}{\vee}$ curves on K3 surfaces.

$C \subseteq X$ K3 $\rightsquigarrow$ C is a (-2)-curve.

E.g. * $\rho(X) \geq 12$ Then, X has a (-2)-curve.

$\uparrow$ Picard number

* $\exists X$ K3 w/ infinitely many (-2)-curves

surfaces

Lem. Given a (-2)-curve C on X,

$\mathcal{O}_C$ is a "spherical object". Seidel

$\rightsquigarrow$ We have a spherical twist $T_C : D^b(X) \xrightarrow{\sim} D^b(X)$.