Only consider varieties over k (separated, integral, finite type)

<u>Defn.</u> A K3 surface is a complete nonsingular variety w/ trivial canonical bundle and irregularity zero. The latter means

$$H^1(X, \mathcal{O}_X) = 0$$

Note we have

$$\Omega_X \otimes \Omega_X \cong \omega_X \cong \mathcal{O}_X$$
$$\varphi \quad \beta \longmapsto \varphi \wedge \beta$$

$\Rightarrow$ that $T_X \cong \Omega_X$.

We can easily generate examples by adjunction. Suppose X is the locus of a degree k homog polynomial in $\mathbb{P}^3$, smooth. By Euler sequence $\omega_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$, so $\omega_{\mathbb{P}^3} \cong \mathcal{O}(-4)$. Adjunction then gives

$$\omega_X \cong \omega_{\mathbb{P}^3} \otimes \mathcal{O}(k)\big|_X \cong \mathcal{O}(k-(n+1))\big|_X$$

so if $k=4$ ω_X is trivial. Take for example

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

We have

$$0 \to \mathcal{O}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

$H^1(\mathbb{P}^3, \mathcal{O}) = 0$ (e.g. by Hodge theory + $\mathbb{P}^3$ simply connected), by Serre duality $H^2(\mathbb{P}^3, \mathcal{O}(-4)) = 0 \Rightarrow H^1(X, \mathcal{O}_X) = 0$. Thus all such surfaces are K3.

A completly analogous adjunction formula calculation gives a complete intersection of type $(d_1,\dots,d_n)$ in $\mathbb{P}^{n+2}$ is K3 <=> $\sum d_i = n+3$.

Only possibilities are $n=1, d_1=4$; $n=2, d_1=2, d_2=3$; $n=3, d_1=d_2=d_3=2$.

Another example are the <u>Kummer surfaces</u>. Let A an abelian surface, the group structure gives an involution

$$\begin{aligned} i : A &\to A \\ x &\mapsto -x \end{aligned}$$

A/i is singular at the 16 two-torsion points. We then blow up these 16 points

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d, dashed] & A \arrow[d] \\ X \arrow[r] & A/i \end{tikzcd}$$

minimal resolution

Arrow exists since i extends to blow up as it leaves blow up points invariant.

Then formulae for blow-ups / branched coverings give

$$\omega_{\tilde{A}} \simeq \mathcal{O}(\textstyle\sum E_i) \simeq \pi^* \omega_X \otimes \mathcal{O}(\textstyle\sum E_i)$$

$\Rightarrow \pi^* \omega_X$ is trivial.

Classical invariants

For a surface we can define the intersection form on $Pic(X)$ as being the intersection number of generic zero sets of sections (w/ multiplicity). More generally there is a map

$$Pic(X) \longrightarrow H^2(X)$$

and we have an intersection form $H^2(X) \times H^2(X) \to H^4(X) \cong \mathbb{C}$

Note if L is ample, its intersection w/ any other line bundle (or curve) is positive.

Riemann-Roch gives

$$\chi(X, L) = \frac{(L . L \otimes \omega_X^*)}{2} + \chi(X, \mathcal{O}_X)$$

The intersection form descends to the Néron Severi group

$$NS(X) := Pic(X)/Pic^0(X)$$

We get $Num(X)$ by further quotienting by numerically trivial bundles.

The Hodge index theorem gives on a Projective variety (generally Kähler) the intersection form on $Num(X)$ is of signature

$$(1, \rho(X) - 1)$$

As a consequence we get for $(L_i)^2 \geq 0$

$$(L_1)^2 (L_2)^2 \leq (L_1 . L_2)^2$$

since $(L_1)^2 L_2 - (L_1 . L_2) L_1$ orthogonal to L_1.

For a K3 surface $h^0(X, \mathcal{O}_X) = 1$, $h^1(X, \mathcal{O}_X) = 0$, $h^2(X, \mathcal{O}_X) \cong H^0(X, \omega_X) = 1$

so $\chi(X, \mathcal{O}_X) = 2$ and R-R formula reads

$$\chi(X, L) = \frac{(L)^2}{2} + 2$$

If L is nontrivial either L or L^* has no global section

Serre duality gives $H^2(X, L) \cong H^0(X, L^*)$, so

$$\chi(X, L) = h^0(X, L) - h^1(X, L)$$

or

$$h^0(X, L^*) - h^1(X, L)$$

For an ample Line bundle L we then have $h^0(X, L^*) = 0$ so

$$-h^1(X, L) = \frac{(L)^2}{2} + 2$$

RHS positive so $h^1(X, L) = 0$.

<u>Prop</u> The natural maps on X a K3

$$\mathrm{Pic}(X) \to NS(X) \to \mathrm{Num}(X)$$

are isomorphisms.

pf/ Suppose L nontrivial, but $(L, L') = 0$ $\forall L'$, in particular for L' ample. We must have $H^0(X, L) = 0$ (else L is trivial or has an effective divisor) and $H^2(X, L) \cong H^0(X, L^*) = 0$ for same reason.

Then $-h^1(X, L) = \frac{(L)^2}{2} + 2$ only possible if $(L)^2 < 0$ so

L cannot be numerically trivial $\square$

In particular there are no torsion line bundles on a K3.

We will show $c_2(X)$ and the Hodge diamond

$$h^{p,q}(X) := \dim H^q(X, \Omega_X^p)$$

are the same for all K3s.

Another version of R-R is

$$2 = \chi(X, \mathcal{O}_X) = \frac{\overbrace{c_1^2(X)}^{0 \text{ since } \omega_X \text{ trivial}} + c_2(X)}{12} = \frac{c_2(X)}{1}$$

$\Rightarrow c_2(X) = 24 \cdot$ orientation class.

By def we have

$$\begin{array}{ccccc} & & 1 & & \\ & ? & & ? & \\ 1 & & ? & & ? \\ & 0 & & ? & \\ & & ? & & \end{array}$$

Symmetries give

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & ? & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

So need to determine $h^{1,1}(X)$. HRR gives this is 20.

Complex K3

A complex K3 is a cpt complex manifold w/ K_X trivial and $H^1(X, \mathcal{O}_X) = 0$.

By GAGA, the algebraic theory is just projective complex K3s. Their are nonprojective K3s, thankfully they're all Kähler. Examples are any non-projective complex torus $\mathbb{C}^2/\Gamma$.

Note we have

$$\overset{0}{H^1(X, \mathcal{O}_X)} \to \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$$

Lefschetz (1,1) says image of this map is

$$H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

a subspace of $H^1(X, \Omega_X)$ which we know is 20-dim.

Thus $\rho(X) \leq 20$.

K3s are examples by Calabi conjecture of Ricci-flat Kähler manifolds.