N. bundles on K3 surfaces.

<u>Mukai vectors</u>

Recall: X sm. proj. var. / $\mathbb{C}$

The Chern character $ch : K^0(-) \to H^{ev}(- ; \mathbb{Q})$ is a unique nat trans. such that $ch : K_0(X) \to H^{ev}(X;\mathbb{Q})$ is a ring hom. sending a line bundle $\mathcal{L}$ to $\exp(c_1(\mathcal{L}))$

(by splitting principle) $\left(ch(\mathcal{F}) = rk + c_1 + \frac{1}{2}(c_1^2 + c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots \right)$

<u>Thm</u> (Hirze.-R-R)

The diagram

$f : X \to pt$

$$\begin{tikzcd} K(X) \arrow[r, "\int_X = f_!"] \arrow[d, "ch"] & K(pt) = \mathbb{Z} \arrow[d, "ch"] \\ H^{ev}(X;\mathbb{Q}) \arrow[r, "\int_X = f_*"'] & H^{ev}(pt;\mathbb{Q}) = \mathbb{Q} \end{tikzcd}$$

commutes after correction by the Todd class $td(X) \in H^{ev}(X;\mathbb{Q})$. ($td(X) = td(T_X)$)

$\left(= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots \right)$

More precisely, for $\mathcal{F}$ coh sheaf on X,

$$\chi(X,\mathcal{F}) = ch\left(\int \mathcal{F}\right) = \int ch(\mathcal{F}) \cdot td(X).$$

$$\chi(X,\mathcal{F}) = \sum (-1)^i \dim_{\mathbb{C}} H^i(X,\mathcal{F}), \qquad \int ch(\mathcal{F}) \cdot td(X) = \sum ch_i(\mathcal{F})\, td_{n-i}(X)$$

We can generalize this to the Euler pairing:

<u>Notation</u>

* For $\mathcal{F}, \mathcal{G} \in Coh(X)$, $\chi(\mathcal{F},\mathcal{G}) := \sum (-1)^i \dim Ext^i(\mathcal{F},\mathcal{G})$. ($\rightsquigarrow \chi(\mathcal{F}) = \chi(\mathcal{O}_X, \mathcal{F})$)
* Define $ch^*(\mathcal{F})$ by $ch_i^* = (-1)^i ch_i$. (if $\mathcal{E}$ is a v.b., $ch^*(\mathcal{E}) = ch(\mathcal{E}^*)$)

$\rightsquigarrow$ HRR: $\chi(\mathcal{F},\mathcal{G}) = \int_X \left(ch^*(\mathcal{F}) \sqrt{td(X)} \right) \cdot \left(ch(\mathcal{G}) \cdot \sqrt{td(X)} \right)$

formal square root $(1+x)^{1/2} = 1 + \frac{1}{2}x - \cdots$

cf.

<u>Def.</u> The **Mukai vector** of $\mathcal{F} \in Coh(X)$ is $v(\mathcal{F}) := ch(\mathcal{F}) \cdot \sqrt{td(X)} \in H^{ev}(X;\mathbb{Q})$.

<u>Rem</u> * We may view $v(\mathcal{F})$ in the Chow ring or the numerical K group.

* $\mathcal{E}$ v.b. and def $v^*(\mathcal{E})$ by $v_i^*(\mathcal{E}) = (-1)^i v_i(\mathcal{E})$.

$\rightsquigarrow v(\mathcal{E}^*) = v(\mathcal{E})^* \cdot \frac{1}{2}c_1(X)$. (FM Lem 5.41).

$\rightsquigarrow \chi(\mathcal{E},\mathcal{F}) = \int_X \frac{1}{2}\exp c_1(X) \cdot v(\mathcal{E})^* \cdot v(\mathcal{F})$

So, if we set $\langle\ ,\ \rangle = \int_X \frac{1}{2} \exp c_1(X) \cdot (-)^* \cdot (-)$ on $H^{ev}(X;\mathbb{Q})$,

then HRR reads $\chi(\mathcal{E},\mathcal{F}) = \langle v(\mathcal{E}), v(\mathcal{F}) \rangle$.

"Mukai pairing", but differs from K3 version by a sign.

Now: X K3 surfaces, $\mathcal{F}$ v. bundle on X,

$$ch(\mathcal{F}) = \operatorname{rk}\mathcal{F} + c_1(\mathcal{F}) + \underbrace{\boxed{\tfrac{1}{2}c_1(\mathcal{F})^2 - c_2(\mathcal{F})}}_{= ch_2(\mathcal{F})}$$

$$td(X) = 1 + \tfrac{1}{2}c_1(X) + \tfrac{1}{12}(c_1^2(X) + c_2(X)) = 1 + \tfrac{1}{12}c_2(X) \rightsquigarrow \sqrt{td(X)} = 1 + \tfrac{1}{24}c_2(X).$$

$$c_1(T_X) = c_1(\Omega_X) = c_1(\omega_X) = c_1(\mathcal{O}_X) = 0$$

$T_X \cong \Omega_X$. $\omega_X = \det \Omega_X$ X K3

HRR: $\chi(\mathcal{F}) = \frac{1}{12}\operatorname{rk}\mathcal{F}\cdot c_2(X) + \boxed{ch_2(\mathcal{F})} = 2\operatorname{rk}\mathcal{F} + ch_2(\mathcal{F})$.

$c_2(X) = 24$ (last time)

$$v(\mathcal{F}) = ch(\mathcal{F})\cdot\left(1 + \tfrac{1}{24}c_2(X)\right) = (\operatorname{rk}\mathcal{F},\ c_1(\mathcal{F}),\ ch_2(\mathcal{F}))\cdot(1, 0, 1)$$

$$= (\operatorname{rk}\mathcal{F}, c_1(\mathcal{F}), ch_2(\mathcal{F}) + \operatorname{rk}\mathcal{F})$$

$$\overset{HRR}{=} (\operatorname{rk}\mathcal{F}, c_1(\mathcal{F}), \chi(\mathcal{F}) - \operatorname{rk}\mathcal{F}) \in \underbrace{H^0 \oplus H^{1,1} \oplus H^4}_{= \widetilde{H}^{1,1}(X)} \subset H^*(X, \mathbb{Z}).$$

E.g. $v(k(x)) = (0, 0, 1)$, $v(\mathcal{O}_X) = (1, 0, \underset{2-1}{1})$, $v(\underset{\text{l.b.}}{\mathcal{L}}) = (1, c_1(\mathcal{L}), \underset{ch_2(\mathcal{F}) + \operatorname{rk}\mathcal{F}}{\tfrac{1}{2}c_1(\mathcal{L})^2 + 1})$.

$$v(T_X) = (2, 0, ch_2(T_X) + 2) = (2, 0, -22)$$

$$ch_2(T_X) = -c_2(T_X) = -24$$

In the case of K3 surfaces (see [FM, p133] in general).

we can write HRR using Mukai vectors.

Def. For a (complex) K3 surface X, the Mukai pairing on $H^*(X, \mathbb{Z})$ is

$$\langle \alpha, \beta \rangle = (\alpha_2.\beta_2) - (\alpha_0.\beta_4) - (\alpha_4.\beta_0),$$

$$= \begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Z}^{22} \\ 0 \\ \mathbb{Z} \end{cases}$$

where $(\,.\,)$ denotes the usual cup product.

(So, $\langle\,,\rangle$ differs from $(\,.\,)$ on $H^0 \oplus H^4$ by signs).

Then HRR reads $\chi(\mathcal{F}, \mathcal{G}) = -\langle v(\mathcal{F}), v(\mathcal{G})\rangle$

E.g. $\chi(T_X, T_X) = -\langle (2, 0, -22), (2, 0, -22)\rangle = 88$

We'll see $\operatorname{Hom}(T_X, T_X) = k$ (and hence $\operatorname{Ext}^2(T_X, T_X) = k$ by Serre duality).

$\longrightarrow \operatorname{Ext}^1(T_X, T_X) = 90$. simple.

Simple bundles X K3

Def. $\mathcal{F} \in \mathrm{Coh}(X)$ is simple if $\mathrm{End}(\mathcal{F}) = k$.

$\rightsquigarrow \chi(\mathcal{F},\mathcal{F}) = 2 - \mathrm{Ext}^1(E,E) \leq 2.$

$\overset{\text{HRR}}{\rightsquigarrow} \langle v(E), v(E)\rangle \geq -2.$

E.g. When $\mathcal{F} = \mathcal{O}_X$ or $\mathcal{O}_C$ (C (-2) curve), $\langle v(E), v(E)\rangle = -2.$

$(\Longleftrightarrow \mathrm{Ext}^1(E,E) = 0)$

Next time: Show T_X is simple. ← only natural non-triv. bundle (that exists on any K3).

In general, we'll use stability.

E.g. (Special case).

Suppose a K3 surface X has $\mathrm{Pic}(X) = 0$.

Then, T_X is simple.

Assume otherwise for contradiction. Then, $\exists\, \varphi \in \mathrm{Hom}(T_X, T_X) \neq k$ that is not an isom.

$\left(\because \text{ pick } \phi \in \mathrm{Hom}(T_X,T_X) \neq k\cdot\mathrm{id},\ x \in X, \text{ and an eigenvalue } \lambda \text{ of } \phi_x : T_X \otimes k(x) \to T_X \otimes k(x). \right.$

$\left. \text{Then, } \varphi := \phi - \lambda\cdot\mathrm{id} \text{ is a desired one} \right).$

Then, φ is not injective since otherwise $\mathrm{cok}(\varphi)$ is a non-triv. tors. sheaf w/ $c_1 = 0$.

$\rightsquigarrow \ker\varphi \neq 0$. On the other hand, $\mathrm{Im}(\varphi) \subseteq T_X$ is torsion-free $\rightsquigarrow \mathrm{pd}\, \mathrm{Im}(\varphi) \leq 1$

Note $\quad 0 \to \ker\varphi \to T_X \to \mathrm{Im}\,\varphi \to 0$

$\underset{\text{LES}}{\rightsquigarrow} \mathcal{E}xt^1(\ker\varphi, -) = \mathcal{E}xt^2(\mathrm{Im}\,\varphi, -) = 0 \quad \rightarrow \ker\varphi$ proj. $\overset{\text{loc. free}}{=}$, $\mathrm{rk} \leq 1$

$\longrightarrow \ker\varphi \cong \mathcal{O}_X$

$\uparrow$

$\mathrm{Pic}(X) = 0$

But then $H^0(X, T_X) \neq 0$, which is absurd (last time). $\square$

Rem. Same arguments show for a non-simple v.b. E, $\exists\, \varphi \in \mathrm{End}(E)$, $\varphi \neq 0$, w/ non-triv. kernel.