Roughly following

- Huybrechts, Chp. 10.1.
- Huybrechts-Lehn, Chp. 4.

Throughout talk, X is a projective variety. $k = \bar{k}$ of char. zero.

Def $\mathcal{C}$ category. Then yon: $\mathcal{C} \hookrightarrow \mathrm{Pre}(\mathcal{C}) =: \mathrm{Functors}(\mathcal{C}^{op}, \mathrm{Set})$.
$y \longmapsto \underline{y}$

Given $\mathcal{F} \in \mathrm{Pre}(\mathcal{C})$, we say that $\mathcal{F}$ is representable if $\mathcal{F} \cong \underline{x}$ for some $x \in \mathcal{C}$.

$\mathcal{F}$ is corepresented by $x \in \mathcal{C}$ if $\exists\, \mathcal{F} \to \underline{x}$, initial among all $y \in \mathcal{C}$.

Equivalently. $\mathcal{F}$ is represented by x if $\forall y \in \mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(y, x) \cong \mathcal{F}(y)$.

$\cdots$ corep $\cdots$ by x $\quad \forall y \in \mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(x, y) \cong \mathrm{Mor}_{\mathrm{Pre}(\mathcal{C})}(\mathcal{F}, \underline{y})$.

x universally corepresents $\mathcal{F}$ if $\forall\, \underline{T} \to \underline{x}$, $\underline{T} \times_{\underline{x}} \mathcal{F}$ corepresented by T.

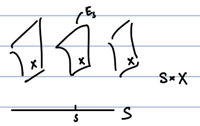
$$\begin{tikzcd} \underline{T} \times_{\underline{x}} \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d] \\ \underline{T} \arrow[r] & \underline{x} \end{tikzcd}$$

RMK: Representable $\Rightarrow$ Corepresentable. When $\mathcal{F}$ is corepresentable, the object corepresenting it is unique up to unique isomorphism.

Def. Let $P \in \mathbb{Q}[z]$. Then let $\mathcal{M} : (\mathrm{Sch}/k)^{op} \to \mathrm{Set}$

Depends on P and $\mathcal{O}_X(1)$.

$$S \longmapsto \{ E \in \mathrm{Coh}(S \times_k X) : E \text{ } S\text{-flat}, \forall s \in S,\ P(E_s) = P,\ E_s \text{ semistable} \} / \sim$$



Theorem $\mathcal{M}$ is corepresented by a projective k-scheme M. The closed pts of M are S-equivalence classes of semistable sheaves on X w/ Hilbert polynomial P.

Why S-equivalence?

1. M is projective. $\quad 0 \to \mathcal{O}(-1) \to E_\lambda \to \mathcal{O}(1) \to 0 \quad \mathbb{P}^1$.

   But $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ and $\mathcal{O} \oplus \mathcal{O}$ are S-equivalent.

2. If E, F are stable, then E, F S-equivalent $\Leftrightarrow E \cong F$.

E^{triv}, E_λ

$E^{triv}|_{\mathbb{A}^1 \setminus \{0\}}$

$$\begin{tikzcd} \mathcal{M}(\mathbb{A}^1) \arrow[r] \arrow[d] & \underline{M}(\mathbb{A}^1) \arrow[d, "\wr"] \\ \mathcal{M}(\mathbb{A}^1 \setminus \{0\}) \arrow[r] & \underline{M}(\mathbb{A}^1 \setminus \{0\}) \end{tikzcd}$$

Strategy for construction: $\quad$ Fix P, X, $\mathcal{O}_X(1)$.

1. There exists $m \in \mathbb{Z}$ such that $\forall$ semistable coherent sheaf F on X w/ Hilbert polynomial P, $F(m)$ is globally generated and $h^0 F(m) = P(m)$.

   - Boundedness results. (Le Potier - Simpson estimates, Grauert-Mülich Theorem, Castelnuovo-Mumford regularity).

   ※ Easy for smooth curves.

   In other words, $H^0(X, F(m)) \otimes \mathcal{O}_X \twoheadrightarrow F(m)$, or $\overbrace{\underbrace{H^0(X, F(m))}_{V} \otimes \mathcal{O}_X(-m)}^{\mathcal{H}} \twoheadrightarrow F$.

2. Consider the quot scheme $\mathrm{Quot}(\mathcal{H}, P)$, which represents the functor $\left(\phi : S \to X \right) \mapsto \left\{ \phi^* \mathcal{H} \twoheadrightarrow E \ \text{ such that } E_s \text{ has Hilbert polynomial } P \right\} / \sim$

   $\quad$ E flat over S, $\mathrm{Supp} E \to S$ is proper

   $$\begin{tikzcd} & \mathrm{Quot}(\mathcal{H}, P) \arrow[d] \\ S \arrow[r, "\phi"'] & X \end{tikzcd}$$

3. Take GIT quotient of the action $GL(V) \curvearrowright \mathrm{Quot}(\mathcal{H}, P)$ with respect to $L_\ell = \det(p_*(\overset{\text{flat}}{\tilde{F}} \otimes q^* \mathcal{O}_X(\ell)))$, which is very ample for large ℓ, and can be $GL(V)$-linearized.

   $\tilde{F}$ is the universal quotient that lives on $\mathrm{Quot} \times X \xrightarrow{p} \mathrm{Quot}$.

   $$\begin{tikzcd} \mathrm{Quot} \times X \arrow[r, "p"] \arrow[d, "q"'] & \mathrm{Quot} \\ X & \end{tikzcd}$$

Brief Review of GIT

$k = \bar{k}$ char 0.

Let G be an algebraic gp $/k$ (finite type). For GIT, want G affine reductive, e.g. $\mathbb{G}_m^n$, PGL_n, SL_n, GL_n.

equivalent to $\subseteq GL_n$ for some n

Right action is a morphism $X \times G \xrightarrow{\sigma} X$ s.t. $\underline{X}(T) \times \underline{G}(T) \to \underline{X}(T)$ is a group action $\forall k$-scheme T.

※ G-equivariant morphism, G-invariant morphism.

Def. (Categorical quotient). $\sigma : X \times G \to X$ gp action. A categorical quotient for σ is a k-scheme Y that corepresents the functor $\underline{X}/\underline{G}$.

The identity morphism $X \to X$ corresponds then to a morphism $X \to Y$ which is G-invariant.

In fact, $\mathrm{Mor}(\underline{X}/\underline{G}, \underline{Y'}) \cong \{ G\text{-invariant maps } X \to Y' \}$.

Def (Good quotient). Let G be an affine alg. gp over k acting on $X \in \mathrm{Sch}/k$.

Then $\varphi : X \to Y$ is a good quotient if

- φ is affine and invariant
- φ is surjective, $U \subset Y$ open $\Leftrightarrow \varphi^{-1} U \subset X$ open.
- $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is iso.
- If W is an invariant closed subset of X, then $\varphi(W)$ is closed in Y.

  If $W_1, W_2 \subset X$ invariant closed disjoint, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.

If a good quotient exists, then denote by $X /\!\!/ G$.

Def. (geometric quotient). A good quotient $X \to Y$ s.t. any geometric fiber is the orbit of a geometric pt is called a geometric quotient.

Theorem (4.2.8 H-L) Let G be reductive acting on $X = \operatorname{Spec} A$ k-scheme of finite type.

Let $Y = \operatorname{Spec} A^G$. Then $X \to Y$ is a universal good quotient.

Example $\mathbb{G}_m \curvearrowright \mathbb{A}^n$. Then the only $k[t,t^{-1}]$-invariant of $k[x_1,\dots,x_n]$ is k. So have universal good quotient $\mathbb{A}^n \to \operatorname{Spec} k$.

This is not geometric because all orbits are collapsed into a single fiber.

Theorem

$$\begin{array}{ccccc} \text{Geometric} & \Rightarrow & \text{Good} & \Rightarrow & \text{Categorical} \\ \Uparrow & & \Uparrow & & \Uparrow \\ \text{Univ. geometric} & \Rightarrow & \text{Univ. good} & \Rightarrow & \text{Universal categorical} \end{array}$$

Proof. Only Good $\Rightarrow$ Categorical is not obvious.

Suppose that $X \to Y$ is a good quotient and $X \to Z$ is invariant. Want map

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\phi"'] & Z \\ Y \arrow[ur, dashed, "\chi"'] & \end{tikzcd}$$

factoring.

Take open affine cover of Z, $\{V_i = \operatorname{Spec} A_i\}$. Then $\psi^{-1}V_i$ is invariant, and so $\psi^{-1}V_i = \phi^{-1}U_i$ for some $U_i \subseteq Y$.

By property of good quotient, U_i is open. So we arrive at the local picture

$$\begin{tikzcd} \phi^{-1}U_i = \psi^{-1}V_i \arrow[r] \arrow[d] & V_i = \operatorname{Spec} A_i \\ U_i \arrow[ur, dashed] & \end{tikzcd}$$

induced by

$$\begin{tikzcd} \Gamma(\psi^{-1}V_i, \mathcal{O}_X) & A_i \arrow[l] \arrow[dl, dashed, "\text{since } A_i \text{ is } G\text{-invariant.}"] \\ \Gamma(U_i, \mathcal{O}_Y) = \Gamma(\phi^{-1}U_i, \mathcal{O}_X)^G \arrow[u] & \end{tikzcd}$$

■

Def (G-linearization) Given coherent sheaf F on $X \circlearrowleft G$, a G-linearization is an isomorphism

$\left(X \times G \overset{p_1}{\underset{\sigma}{\rightrightarrows}} X\right)$ $\qquad \sigma^* F \xrightarrow[\cong]{\Phi} p_1^* F$, such that certain cocycle conditions are satisfied.

Fiberwise, isomorphisms $F_{gx} \cong F_x$ $\forall x \in X$.

$$x, g_1, g_2 \longmapsto x g_1 g_2$$

$$\begin{tikzcd} & & X \times G \times G \arrow[dl, "\sigma \times id_G"'] \arrow[d, "p_{12}"] \arrow[r, "id \times \mu"] & X \times G \arrow[r, "\sigma"] \arrow[d, "p_1"] & X \\ & X \times G \arrow[dl, "\sigma"'] \arrow[d, "p_1"] & X \times G \arrow[d, "\sigma"] \arrow[dr, "p_1"] & X & \\ X & X & X & X & \end{tikzcd}$$

$$\begin{array}{cccc} xg_1g_2 & xg_1 & xg_1 & x \end{array} \qquad x$$

$$\begin{tikzcd} (\sigma \times id_G)^* \sigma^* F \arrow[d, equal] & p_{12}^* \sigma^* F \arrow[d, equal] & \\ (id \times \mu)^* \sigma^* F \arrow[r, "(\sigma \times id_G)^* \Phi"] \arrow[rr, bend right, "(id \times \mu)^* \Phi"'] & (\sigma \times id_G)^* p_1^* F \arrow[r, "p_{12}^* \Phi"'] & p_{12}^* p_1^* F \arrow[d, equal] \\ & & (id \times \mu)^* p_1^* F \end{tikzcd}$$

Remark: The space of global sections of a G-linearized qcoh sheaf naturally is a G-representation.

$\Rightarrow$ If L is very ample G-linearized, have that $X \longrightarrow \mathbb{P}(H^0(X, L))$ is G-equivariant.

So that we've "linearized" the G-action on X into G-action on a vector space / projective space.

Taking quotients of a projective variety by a reductive group:

Suppose $X \circlearrowleft G$, L G-linearized ample line bundle, then

$$R = \bigoplus_{n \geq 0} H^0(X, L^{\otimes n})$$

is a f.g. $\mathbb{Z}$-graded k-algebra w/ G-action respecting the grading.

Then R^G is also f.g. $\mathbb{Z}$-graded k-algebra, and the inclusion $R^G \hookrightarrow R$ induces

$$\begin{tikzcd} \operatorname{Proj} R \arrow[r, dashed] & \operatorname{Proj} R^G \\ U = \operatorname{Proj} R \setminus V(R_+^G \cdot R) \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}.$$

A point $x \in \operatorname{Proj} R$ lies in U iff $\exists s \in H^0(X, L^{\otimes n})^G$ for some n such that $s(x) \neq 0$.

Let $X = \operatorname{Proj} R$. $Y = \operatorname{Proj} R^G$.

Definition A point $x \in X$ is semistable (with respect to L) if $x \notin V(R_+^G \cdot R)$. This open set is denoted $X^{ss} = X^{ss}(L)$.

- - - - - - stable if G_x is finite and the G-orbit of x is closed in X^{ss}. This is also an open condition, denoted $\underset{\substack{\shortparallel \\ X^s(L)}}{X^s} \subset X^{ss}$.

Theorem Let G be reductive $\curvearrowright$ projective scheme X w/ a G-linearized ample line bundle L.

Then $\exists$ projective scheme Y and a morphism $\pi: X^{ss} \to Y$ which is a universal good quotient.

Moreover, $\exists$ open $Y^s \subset Y$ s.t. $X^s(L) = \pi^{-1} Y^s$ and $X^s \to Y^s$ is a universal geometric quotient.

Lastly, we can take for some positive m, a very ample line bundle M on Y s.t. $L^{\otimes m}|_{X^{ss}(L)} = \pi^* M$.

Lastly, a useful criterion to determine stability / semistability with respect to some L:

Theorem (Hilbert–Mumford criterion).

Definition Invariant function means?

char k = 0.

Theorem 1.1. Let X be an affine scheme over k, let G be a reductive algebraic group, and let $\sigma : G \times X \to X$ be an action of G on X. Then a ~~uniform~~ universal categorical quotient (Y, ϕ) of X by G exists, ϕ is universally submersive, and Y is an affine scheme. Moreover, if X is algebraic, then Y is algebraic over k.

~~If char. $(k) = 0$, (Y, ϕ) is a universal categorical quotient. Moreover~~

✱ X noetherian implies Y noetherian.

✱ Separates disjoint closed invariant subschemes.

Theorem 1.10. Let X be an algebraic pre-scheme over k, and let G be a reductive algebraic group acting on X. Suppose L is a G-linearized invertible sheaf on X. Then a ~~uniform~~ universal categorical quotient (Y, ϕ) of $X^{ss}(L)$ by G exists. Moreover:

(i) ϕ is affine and universally submersive;

(ii) there is an ample invertible sheaf M on Y such that $\phi^*(M) \cong L^n$ for some n; hence Y is a quasi-projective algebraic scheme;

(iii) there is an open subset $\tilde{Y} \subset Y$ such that $X^s(L) = \phi^{-1}(\tilde{Y})$ and such that $\left(\tilde{Y}, \phi \mid X^s(L)\right)$ is a uniform geometric quotient of $X^s(L)$ by G.

Definition 1.7. Let x be a geometric point of X. Then:

(a) x is *pre-stable* (with respect to σ) if there exists an invariant affine open subset $U \subset X$ such that x is a point of U, and the action of G on U is closed. orbits of geometric pts are closed

Now suppose L is an invertible sheaf on X, and ϕ is a G-linearization of L. Then:

(b) x is *semi-stable* (with respect to σ, L, ϕ) if there exists a section $s \in H^0(X, L^n)$ for some n, such that $s(x) \neq 0$, X_s is affine, and s is invariant, i.e. if $\phi_n : \sigma^*(L^n) \to p_2^*(L^n)$ is induced by ϕ, then $\phi_n(\sigma^*(s)) = p_2^*(s)$.

(c) x is *stable* (with respect to σ, L, ϕ) if there exists a section $s \in H^0(X, L^n)$ for some n, such that $s(x) \neq 0$, X_s is affine, s is invariant, and the action of G on X_s is closed.