K3

$k = \bar{k}$

Defn Fix a Mukai vector $v = (r, \ell, s) \in H^*(X, \mathbb{Z})$

Define $M_v : Sch_k \to Sets$

$$\begin{matrix} T \\ \downarrow \\ \operatorname{Spec} k \end{matrix} \mapsto \left\{ \mathcal{E} \in coh(X \times_k T) \;\middle|\; \mathcal{E} \text{ is flat } /T,\ \underset{\mathcal{E}_t \text{ semistable}}{v(\mathcal{E}_t) = v} \ \forall t \in T \right\} \Big/ \sim$$

$\mathcal{E} \sim \mathcal{E}'$ if $\mathcal{E} \simeq \mathcal{E}' \otimes p^* \mathcal{L}$

$X \times T \xrightarrow{p} T$

$\mathcal{L} \in Pic(T)$

Fact:

M_v is a subfunctor of M_h since $P_E(m) = \chi(E(m)) = -\langle v(E), v(\mathcal{O}(-m)) \rangle$

Fix some polynomial h

$Sch_k \to Sets$

$$\begin{matrix} T \\ \downarrow \\ \operatorname{Spec} k \end{matrix} \mapsto \left\{ \mathcal{E} \in coh(X \times_k T) \;\middle|\; \mathcal{E} \text{ is flat } /T,\ \mathcal{E}_t \text{ semistable and } P_{\mathcal{E}_t} = h \ \forall t \in T \right\}$$

Rmk: M_v makes sense bc when $\mathcal{E}$ is flat $/T$, $\chi(E_t, F)$ is constant $\forall F$

$$\chi(E_t, F) = \langle v(E_t), v(F) \rangle.$$

Local structure of M_v

- if M_v is a fine moduli space (ie. $M_v \simeq h_M$), then $\boxed{T_t M \cong Ext^1(E, E)}$ where $E \in coh(X)$ corresponds to $t \in M_v(k)$.

reason:

$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & X_{k[\varepsilon]/\varepsilon^2} \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & X_M \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2} \arrow[r, "\phi"] & M \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$\frac{k[\varepsilon]}{\varepsilon^2} \twoheadrightarrow k$

Fix $t \in M(k)$ that corresponds to E

$$\Phi : \operatorname{Hom}_k\left(\operatorname{Spec} \tfrac{k[\varepsilon]}{\varepsilon^2}, M\right)_t \to Ext^1(E, E)$$

$$\operatorname{Hom}_k\left(\operatorname{Spec} \tfrac{k[\varepsilon]}{\varepsilon^2}, M\right)_t \cong T_t M \qquad Ext^1(E, E) \cong \{\text{Extensions } 0 \to E \to \mathcal{F} \to E \to 0\}$$

pull back then $\otimes$ $\quad 0 \to k \to \frac{k[\varepsilon]}{\varepsilon^2} \to k \to 0$

$$\phi \mapsto \mathcal{F} \in coh(X_{k[\varepsilon]/\varepsilon^2}) \mapsto (0 \to \mathcal{F}|_k \to \mathcal{F} \to \mathcal{F}|_k \to 0) \text{ on } X_{k[\varepsilon]/\varepsilon^2}$$

(note $\mathcal{F}$ flat $/ \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2}$).

$(\mathcal{F} \in M(\frac{k[\varepsilon]}{\varepsilon^2}))$

$\downarrow$ push forward

$(0 \to E \to f_* \mathcal{F} \to E \to 0)$ on X.

inverse:

$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & X_{k[\varepsilon]/\varepsilon^2} \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & X_M \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[r, "\phi"'] & M \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$k[\varepsilon]/\varepsilon^2 \twoheadrightarrow k$

$0 \to E \to \mathcal{F} \to E \to 0$ on X.

$\nearrow$ can give $\mathcal{F}$ a $\underline{\mathcal{O}_X} + \underset{\Delta}{\varepsilon\mathcal{O}_X}$ module structure

$\cdot\varepsilon : \mathcal{F} \to E \to \mathcal{F}$

$\Rightarrow$ Can define $\mathcal{F}$ as on $X_{k[\varepsilon]/\varepsilon^2}$

(can check that $\mathcal{F} \rightsquigarrow \operatorname{Hom}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, M)_t$.

- If M_v is a coarse moduli space.

<u>Prop</u>. (1.11 in K3 H).

Let M be the moduli space of $\mathcal{M}_v$. and let $t \in M$ be a pt corresponding to a stable sheaf $E \in \mathcal{M}(k)$.

(i). Then there exists a natural iso

$$T_t M \simeq \operatorname{Ext}^1(E, E).$$

(ii). If $\operatorname{Ext}^2(E, E)$ is 0, then M is smooth at $t \in M$.

(iii). If the trace map $\operatorname{Ext}^2(E, E) \to H^2(X, \mathcal{O})$ is injective and Pic_X is smooth at the pt corresponding to $\det(E)$, then M is smooth at $t \in M$.

$$\operatorname{Ext}^2(E, E) \times \operatorname{Hom}(E, E) \longrightarrow \operatorname{Ext}^2(E, E) \xrightarrow{tr} H^2(X, \mathcal{O}) \simeq k$$

$$\operatorname{Ext}^2(E \otimes E^\vee, \mathcal{O}_X) \times \operatorname{Hom}(\mathcal{O}, E \otimes E^\vee).$$