


K3



$k = \mathbb{K}$

Defn: Fix a Mukai vector $v = (r, c, s) \in H^*(X, \mathbb{Z})$

Define $M_v : \text{Sch}_k \rightarrow \text{Sets}$

$$\begin{array}{c} T \\ \downarrow \\ \text{Spec } k \end{array} \mapsto \left\{ E \in \text{coh}(X \times_k T) \mid \begin{array}{l} E|_T \text{ is flat} \\ E|_T \text{ is } v\text{-flat} \\ \forall t \in T \end{array} \right\}$$

$E|_T$ semistable
 $E \in \mathcal{E}'$
 $E = \mathcal{E}' \otimes \mathcal{P}^* \otimes \mathcal{L}$
 $X \times T \xrightarrow{\mathcal{P}} T \xrightarrow{\mathcal{L}} \text{pt}$

Fact: M_v is a subfunctor of M_h since $P_E(m) = \chi(E(m)) = -\langle v(E), v(O(-m)) \rangle$

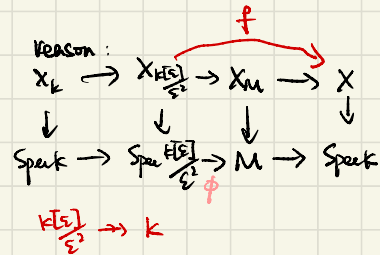
Fix some polynomial h
 $\text{Sch}_k \rightarrow \text{Sets}$

$$\begin{array}{c} T \\ \downarrow \\ \text{Spec } k \end{array} \mapsto \left\{ E \in \text{coh}(X \times_k T) \mid \begin{array}{l} E|_T \text{ is flat} \\ E|_T \text{ is } v\text{-flat} \\ P_E = h \end{array} \right\}$$

Remark: M_v makes sense bc when E is flat / T , $\chi(E_t, F)$ is constant
 $\langle v(E_t), v(F) \rangle$

Local structure of M_v

- if M_v is a fine moduli space (ie $M_v \cong h_M$) then $T_t M \cong \text{Ext}^1(E, E)$
 where $E \in \text{coh}(X)$ corresponds to $t \in M(k)$



Fix $t \in M(k)$ that corresponds to E
 $\Phi: \text{Hom}_k(\text{Spec } \frac{\mathbb{K}[t]}{\mathbb{K}}, M)_t \rightarrow \text{Ext}^1(E, E)$

$T_t M$ Extensions $0 \rightarrow E \rightarrow F \rightarrow E \rightarrow 0$

pull back then ϕ $0 \rightarrow k \rightarrow \mathbb{K}[t] \rightarrow k \rightarrow 0$

$$\phi \mapsto F \in \text{coh}(X_{\mathbb{K}[t]/\mathbb{K}}) \mapsto (0 \rightarrow F|_k \rightarrow F \rightarrow F|_k \rightarrow 0)$$

(note F flat / $\text{Spec } \frac{\mathbb{K}[t]}{\mathbb{K}}$)
 $(F \in M(\frac{\mathbb{K}[t]}{\mathbb{K}}))$

push forward

$$(0 \rightarrow E \rightarrow F \rightarrow E \rightarrow 0)$$

on X

inverse:

$$\begin{array}{ccccccc}
 X_k & \rightarrow & X_{\frac{HSJ}{\mathcal{E}}} & \rightarrow & X_M & \rightarrow & X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Spec } k & \rightarrow & \text{Spec } \frac{HSJ}{\mathcal{E}} & \rightarrow & M & \rightarrow & \text{Spec } k \\
 & & \downarrow \phi & & & & \\
 \frac{HSJ}{\mathcal{E}} & \rightarrow & k & & & &
 \end{array}$$

f

$$0 \rightarrow E \rightarrow \mathcal{F} \rightarrow E \rightarrow 0 \quad \text{on } X$$

can give \mathcal{F} a $\mathcal{O}_X + \mathcal{E}\mathcal{O}_X$ module structure

$\cdot \mathcal{E}: \mathcal{F} \rightarrow E \rightarrow \mathcal{F}$

\Rightarrow Can define \mathcal{F} as on $X_{\frac{HSJ}{\mathcal{E}}}$

(can check that $\mathcal{F} \rightarrow \text{Hom}(\text{Spec } \frac{HSJ}{\mathcal{E}}, M)_+$)

- If M_V is a coarse moduli space.

Prop. (1.11 in K3 H).

Let M be the moduli space of M_V and let $t \in M$ be a pt corresponding to a stable sheaf $E \in M(k)$.

(i). Then there exists a natural iso

$$T_t M \simeq \text{Ext}^1(E, E).$$

(i'). If $\text{Ext}^1(E, E)$ is 0, then M is smooth at $t \in M$.

(ii'). If the trace map $\text{Ext}^2(E, E) \rightarrow H^2(X, \mathcal{O})$ is injective and Pic_X is smooth at the pt corresponding to $\det(E)$, then M is smooth at $t \in M$.

$$\text{Ext}^2(E, E) \times \text{Hom}(E, E) \rightarrow \text{Ext}^2(E, E) \xrightarrow{\text{tr}} H^2(X, \mathcal{O}) \simeq k$$

$$\text{Ext}^2(E \otimes E^\vee, \mathcal{O}_X) \times \text{Hom}(\mathcal{O}, E \otimes E^\vee).$$