$$\text{E.g.}\quad X \text{ surface} \overset{\text{ample div. } H}{\curvearrowleft},\quad \mathcal{F} \in \mathrm{Coh}(X)$$

On a surface, $P(\mathcal{F}, m) = \dfrac{\underline{\mathrm{rk}(\mathcal{F}) \cdot (H)^2}}{2} m^2 + (\underset{\deg_H \mathcal{F}}{\underbrace{H \cdot c_1(\mathcal{F})}} + \mathrm{rk}(\mathcal{F}) \underset{\deg_H (T_X)}{\underbrace{(H \cdot c_1(X))}}) m + \alpha_0(\mathcal{F})$ by HRR.

(i) $\dim \mathcal{F} := \dim \operatorname{supp} \mathcal{F} = 0 \rightsquigarrow P(\mathcal{F}, m) = \text{const} \longrightarrow p(\mathcal{F}, m) = 1.$

Clearly, $\mathcal{F}$ is pure and semi-stable.

$\mathcal{F}$ is stable $\Leftrightarrow$ $\mathcal{F} = k(x)$ for some $x \in X_k$.

(ii) $\dim \mathcal{F} = 1 \rightsquigarrow P(\mathcal{F}, m) = \deg_H \mathcal{F} \cdot m + \alpha_0(\mathcal{F})$
$(\mathrm{rk}\,\mathcal{F} = 0)$

$\mathcal{F}$ is stable $\Leftrightarrow$ $\mathcal{F}$ pure and $\forall\, 0 \neq \mathcal{G} \subsetneq \mathcal{F}$,

$$\begin{cases} \deg_H \mathcal{F} > \deg_H \mathcal{G} \\ \deg_H \mathcal{F} = \deg_H \mathcal{G}, \quad \alpha_0(\mathcal{F}) > \alpha_0(\mathcal{G}). \end{cases}$$

If $\mathcal{F}$ is supported on an int. curve $C \subset X$,

then $\mathcal{F}$ is stable $\Leftrightarrow$ $\mathcal{F}|_C$ is μ-stable.

(iii) $\dim \mathcal{F} = 2 \mapsto \mathcal{F}$ is stable $\Leftrightarrow$ $\mathcal{F}$ is tors.-free and

$$p(\mathcal{F}, m) = m^2 + \left(\frac{\deg_H(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} + \deg_H(T_X)\right) \frac{m}{(H)^2/2} + \frac{\alpha_0(\mathcal{F})}{\mathrm{rk}\,\mathcal{F} \cdot (H)^2/2}$$

$\forall\, 0 \subsetneq \mathcal{G} \subsetneq \mathcal{F}$,

$$\begin{cases} * \ \underset{\mu_H(\mathcal{F})}{\underbrace{\dfrac{\deg_H(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}}} > \dfrac{\deg_H(\mathcal{G})}{\mathrm{rk}(\mathcal{G})} = \mu_H(\mathcal{G}) \\ * \ \mu_H(\mathcal{F}) = \mu_H(\mathcal{G}), \quad \dfrac{\alpha_0(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} > \dfrac{\alpha_0(\mathcal{G})}{\mathrm{rk}(\mathcal{G})} \end{cases}$$

Cor. X surface, $\mathcal{F}$ tor. free

μ-stable $\Rightarrow$ stable $\Rightarrow$ semi-stable $\Rightarrow$ μ-semi-stable.

X proj. var.

Called a Jordan–Hölder filtration

Prop. Let E be a semistable sheaf. Then, $\exists$ a filtration

$$0 \subset E_0 \subset \cdots \subset E_n = E$$

s.t. $\forall$ quot. E_{i+1}/E_i are stable w/ reduce Hib. poly. $p(E, m)$.

The isom. class of the graded obj.

$$JE(E) := \bigoplus E_{i+1}/E_i$$

is indep. of the filtration.

Def'n. Two semistable sheaves E and F are S-equalve :iff $JH(E) \simeq JH(F)$.