Notation

(X, H) polarized algebraic K3 (resp. (X, ω) Kähler K3)

$\deg_H E := (c_1(E) . H)$ (resp. $\deg_\omega(E) := (c_1(E) . \omega)$.

Rem. $(X, H) \rightsquigarrow (X, \omega_{FS}|_X)$ w/ $c_1(H) = \omega_{FS}|_X$ $(\because c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = \omega_{FS}) \rightsquigarrow \deg_H = \deg_{\omega_{FS}|_X}$

From now on, we don't specify H or ω, but deg depends on them.

Def. Suppose $\mathrm{rk}(E) \neq 0$.

Define the slope of E to be $\mu(E) := \dfrac{\deg E}{\mathrm{rk}(E)}$

A torsion free sheaf E is called μ-stable (or slope stable) (resp. μ-semistable)
if for any subsheaf $F \subset E$ w/ $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$,
we have $\mu(F) < \mu(E)$
(resp. $\leq$).

Lem. X smth proj. var.

(i) Any line bundle is μ-stable.

(ii) For a SES $0 \to F \to E \to G \to 0$ w/ $\mathrm{rk}(F), \mathrm{rk}(G) \neq 0$,

$\mu(F) < \mu(E) \iff \mu(E) < \mu(G)$.

Cor. If E is loc. free, E is μ-stable
X smth surface
$\iff$ for any loc. free $F \subset E$, $\mu(F) < \mu(E)$
w/ $\mathrm{rk} F < \mathrm{rk} E$
and tors. free quot.

(iii) μ-stable $\Rightarrow$ simple

(iv) E is μ-stable $\iff$ E^* is μ-stable $\iff$ E^{**} is μ-stable
loc. free and
$\mu(E) = \mu(E^{**})$.

Proof.) (i) vacant

(ii) Note $\deg E = \deg F + \deg G$ and $\mathrm{rk}\, E = \mathrm{rk}\, F + \mathrm{rk}\, G$.

Proof of Cor.)

By (ii), F is μ-stable if $^{\forall} F \subset E$ w/ $0 < \mathrm{rk}\, F < \mathrm{rk}\, E$, $\mu(E) < \mu(E/F)$

$0 \leftarrow (E/F)/T(E/F) \twoheadleftarrow E \leftarrow \ker \hookleftarrow 0$

Since $^{\forall}H$, $\mu(H/T(H)) < \mu(H)$. we may assume E/F is torsion free $\rightsquigarrow$ F is loc. free by the following lem.

<u>Lem.</u> $0 \to F \to E \to G \to 0$ on a smooth surface X. If E is loc. free and G is torsion free, then F is loc. free

proof). Take $x \in X$.

<u>Def.</u> $\operatorname{depth} M = \min\{i \mid \operatorname{Ext}^i(k(x), M) \neq 0\}$ ($\mathcal{O}_x$-mod) $=$ max lgth. of reg. seq.

$$0 \to F_x \to E_x \to G_x \to 0$$

$E_x \cong \mathcal{O}_x^{\oplus n}$, G_x tors. free.

Recall $\operatorname{pd} F_x + \operatorname{depth} F_x = \operatorname{depth} \mathcal{O}_x = \dim \mathcal{O}_x = 2$, ($X$ regular ($\Rightarrow$ Cohen-Mac), X surface)

so it suffices to see $\operatorname{depth} F_x = 2$ ($\rightsquigarrow \operatorname{pd} F_x = 0$).

Indeed, since G_x is tors. free $\rightsquigarrow$ $^{\forall}a \in \mathcal{O}_x$ is G_x-regular.

$\to \operatorname{depth} G_x \geq 1$.

$\to \operatorname{depth} F_x \geq 2$. $\square$

(iii) Suppose E is not simple.

Then, $^{\exists}\psi \in \operatorname{Hom}(E, E) \setminus k$ w/ non-triv. kernel ($\rightsquigarrow \underset{\wedge 0}{\mathrm{rk}\ker} + \underset{\wedge 0,\ \text{tors. free}}{\mathrm{rk}\,\mathrm{Im}\,\psi} = \mathrm{rk}\, E$).

($\because$ pick $^{\#}\phi \in \operatorname{Hom}(E, E)$ ($\neq k\cdot \mathrm{id}$), $x \in X$, and an eigenvalue λ of $\phi_x : T_x \otimes k(x) \to T_x \otimes k(x)$.
Then, $\psi := \phi - \lambda \cdot \mathrm{id}$ is a desired one).

Hence, $0 < \mathrm{rk}\,\mathrm{Im}\,\psi < \mathrm{rk}\, E$. On the other hand, since E is μ-stable, $\mathrm{Im}\,\psi \subset E$ and $E \twoheadrightarrow \mathrm{Im}\,\psi$ imply $\mu(\mathrm{Im}\,\psi) < \mu(E) < \mu(\mathrm{Im}\,\psi)$!!

(iv) Corollary of (ii)

E.g. We can produce stable bundles from certain l. bundles on K3.

(i) $\mathcal{L}$ globally generated, ample

Then, $\ker(H^0(X,\mathcal{L})\otimes\mathcal{O}_X \to \mathcal{L})$ is μ-stable

(ii) $\mathcal{L}$ globally generated and generates $\mathrm{Pic}(X)$.

Then, $\ker(H^0(X,\mathcal{L})\otimes\mathcal{O}_X \to \mathcal{L})$ is μ-stable.

Now, we'll show T_X on a complex K3 is μ-stable.

Note T_X is μ-stable $\Leftrightarrow$ $\forall$ l.b. $\mathcal{L}\subset T_X$ w/ tors. free quot., $\mu(\mathcal{L})=\deg\mathcal{L}<0=\mu(T_X)$.

E.g. If $\mathrm{Pic}(X)=0$, then the only possible testing bundle for T_X is $\mathcal{O}_X$. However, $H^0(X,T_X)=0$ by Hodge theory, so it's also impossible. So, T_X is vacantly μ-stable.

* Algebraic approach.

Thm. (X,H) polarized K3 over an alg. closed field k of char. zero. If a l.b. $\mathcal{L}\subseteq T_X$ has the tors. free quot. w/ $\deg_H\mathcal{L}>0$, then for a generic $x\in X$, $\exists$ rat. curve $x\in C\subset X$ s.t. $T_C(x)\subset\mathcal{L}(x)\subset T_X(x)$

Cor. (X,H) as above. T_X does not contain any l.b. of positive degree w/ tor. free quot.

proof) "K3 surface cannot contain too many rational curves".

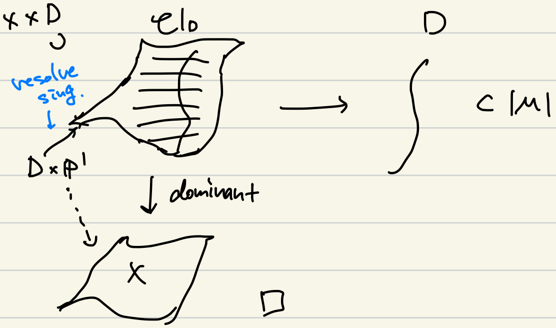
Assume $\exists\ \mathcal{L}\subset T_X$ w/ $\deg\mathcal{L}>0$ and tor. free quot.

WMA k is uncountable by base change. Then, $\forall\, U\overset{\text{open}}{\subset} X$, $U(k)$ cannot be covered by countably many curves. On the other hand, the theorem says $\exists\, U\overset{\text{open}}{\subset} X$ that is covered by a rational curves.

Claim. X is uniruled, i.e., $\exists$ rat. dom. map $D\times\mathbb{P}^1 \dashrightarrow X$ for some D.

Idea of proof). Since $\mathrm{Pic}(X)\simeq NS(X)$ is countable, this implies $\exists$ linear system $|M|$ containing uncountably many rational curves.

We may view $|M|$ as a moduli space w/ univ. family $\mathcal{C} \subseteq X \times |M|$.
Moreover, since being rational is closed condition in $|M|$, there exists a curve $D \subseteq |M|$ whose fibers in X are rational curves. Consider the restriction of $\mathcal{C}$ to D:



$\square$

(cf. Def. 4.1. and Prop. 4.12 for general argument)

Claim. K3 surface X over an alg. closed field of char. 0 is not uniruled.

proof.) Assume X is uniruled.

By resolving indeterminancy of $D \times \mathbb{P}^1 \dashrightarrow X$,

we get dominant $Y \to X$, which is generically étale (by generic smoothness).

$H^0(U_X, \omega|_U) \hookrightarrow H^0(U_Y, \omega_Y|_U)$ $\quad f^*\omega_X|_{U_X} \cong \omega_Y|_{U_Y}$ $\quad$ or EGA IV_2 Cor. 2.2.8.

Then, $H^0(X, \omega_X) \to H^0(Y, \omega_Y)$ is injective. WMA f surj. $\Rightarrow$ f faith. flat

On the other hand, $H^0(X, \omega_X) \neq 0$ while $h^0(Y, \omega_Y) = h^0(\mathbb{P}^1 \times D, \omega_{\mathbb{P}^1 \times D})$
$= h^0(\mathbb{P}^1, \omega) \times h^0(D, \omega_D)$
$= 0$ $\square$

Cor. (X, H) as above. Then, T_X is μ-stable.

proof). Suffices to test w/ l.b. $\mathcal{L} \subset T_X$ w/ tors. free quot.
By Cor. above, $\deg_H \mathcal{L} \le 0$. Assume $\deg_H \mathcal{L} = 0$. If $\mathcal{L} \ncong \mathcal{O}$, then $\exists H'$ s.t. $\deg_{H'} \mathcal{L} > 0$, which is absurd. So, $\mathcal{L} = \mathcal{O}$.
However, since $H^0(X, T_X) = 0$ by Hodge theory, it is also absurd.
Thus, $\deg_H \mathcal{L} < 0$. $\square$